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Large amplitude waves in stratified media: acoustic pulses

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A general theory is presented which describes disturbances generated by large amplitude, high frequency pulses in stratified media. The theory is used to discuss large amplitude, but shockless, acoustic pulses propagating under the influence of a constant gravity force into an atmosphere which need not be in thermal equilibrium before their arrival.

The modulating influences of pressure and temperature stratification on the amplitudes of such pulses as they move towards or away from earth are described in detail.

The paper generalizes techniques already established by Whitham (1956) so that they are applicable to disturbances of any amplitude.

1. Introduction

In this and subsequent papers a theory is presented which describes the propagation of large amplitude, high frequency waves in stratified media. This, the first paper, neglects all phenomena associated with the geometry of such waves. It deals with plane pulses in which conditions are governed by solutions $\mathbf{u}(t, X) = (u_1, u_2, \ldots, u_n)$ to a system of hyperbolic equations which can be written in the form

$$\mathbf{u}_{,X} + \mathbf{A}(\mathbf{u}, X)\,\mathbf{u}_{,t} = 0. \tag{1.1}$$

As an illustration of the theory, conditions in a large amplitude, non-isentropic, acoustic-gravity pulse, which may be moving towards or away from earth, are discussed in detail. The entropy variation in the pulse is caused by ambient thermal and density stratification before its arrival.

Although equations such as (1.1) govern a wide variety of wave-like disturbances in stratified media, few results which have any general applicability are known. Even when (1.1) are formally linearized about some constant ambient state, $\mathbf{u} = 0$ say, so that $\mathbf{A}$ is approximated by $\mathbf{A}_0(X) = \mathbf{A}(0, X)$, the only results of a general nature have been obtained in the geometrical acoustics limit. Then, the stratification, which results from the dependence of $\mathbf{A}_0$ on X, is 'slowly varying' for the disturbance. If $w_0(X)$ is any eigenvalue of $\mathbf{A}_0(X)$ and if

$$\alpha = t - \int_0^X w_0(s)\,ds \tag{1.2}$$

denotes the associated characteristic variable, then the linear equations have regular asymptotic expansions of the form

$$\mathbf{u} = \sum_{m=0}^{\infty} \tau^m f_m(\beta)\,\hat{\mathbf{U}}_m(\alpha, X) = \hat{\mathbf{U}}(\alpha, X\!:\tau) \quad \text{say.} \tag{1.3}$$

In (1.3) the small parameter τ is typically the ratio of a time scale introduced by the boundary data to a time scale defined by the stratification. The f_m are functions of the fast characteristic variable

$$\beta = \alpha/\tau. \tag{1.4}$$

They satisfy the recurrence relations

$$df_{m+1}/d\beta = f_m \quad (m = 0, 1, \ldots). \tag{1.5}$$

The $\hat{\mathbf{U}}_m(\alpha, X)$ are functions of X and the slow characteristic variable α. They satisfy the recurrence relations

$$(\mathbf{A}_0(X) - w_0(X)\,\mathbf{1})\,(\hat{\mathbf{U}}_{m+1} - \hat{\mathbf{U}}_{m,\alpha}) = -\hat{\mathbf{U}}_{m,X} \quad (m = 0, 1, \ldots). \tag{1.6}$$

In (1.5) the signal function $f_0(\beta)$ is arbitrary; in (1.6)

$$\hat{\mathbf{U}}_0 = \sigma(\alpha, X)\,\mathbf{r}_0(X), \tag{1.7}$$

where $\mathbf{r}_0(X)$ is any right eigenvector of $\mathbf{A}_0(X)$ corresponding to the eigenvalue $w_0(X)$. The scalar $\sigma(\alpha, X)$ satisfies the transport equation

$$\sigma_{,X} + K(X)\,\sigma = 0, \tag{1.8}$$

where $K(X)$ is given in terms of $\mathbf{r}_0(X)$ and any associated left eigenvector $\mathbf{l}_0(X)$ by

$$K = \mathbf{l}_0 \cdot \frac{d\mathbf{r}_0}{dX} \Big/ \mathbf{l}_0 \cdot \mathbf{r}_0. \tag{1.9}$$

The solutions described by (1.2)–(1.9) are directly applicable in regions where only one of the components of the disturbance, that associated with the family of characteristics given by $\alpha(t, X) =$ constant, is excited. They describe disturbances which are of high frequency in the sense that the signals carried by the progressing waves which generate them are *sharp*: the main effect of the stratification is to attenuate or amplify the waves rather than to distort, or disperse, them. For, according to (1·3), to a first approximation conditions at a station X at the arrival of the wavelet $\alpha =$ constant, which left $X = 0$ at $t = \alpha$, are determined by conditions at $X = 0$ at $t = \alpha$ and are independent of conditions at all precursor wavelets. The general aim of this study is to construct a theory which describes the dominant behaviours of large amplitude progressing waves in this high frequency limit.

Of course expansions of the form (1.3) are out of the question for non-linear systems. However, there is an alternate interpretation of these expansions which immediately suggests a procedure for tackling the non-linear problem. The waves described by these expansions are examples of waves in which all the components of $\mathbf{u}$ are relatively undistorted. Quite loosely, the variables $\mathbf{f}(t, X) = (f_1, f_2, \ldots, f_m)$ are relatively undistorted in a wave if at all (t, X) of interest there exists a propagating surface $\alpha(t, X) =$ constant, called a wavelet, such that the magnitude of the rate of change of any of the f_i $(i = 1, 2, \ldots, m)$, moving with the wavelet is small compared with the magnitude of the rate of change of f_i at fixed t or X. The u_i $(i = 1, 2, \ldots, n)$, represented by (1.3), with α given by (1.2), are relatively undistorted because, in the terminology introduced in (1.3),

$$U_{i,X} = O(\tau)\, u_{i,X}. \tag{1.10}$$

The definition of a relatively undistorted wave given above does not restrict the amplitude of the wave to be small. Evidence that there are such waves of any amplitude is furnished by the following observation. If $\mathbf{A}$ in (1.1) is independent of X then any progressing wave is a simple wave. In such a wave the components of $\mathbf{u}$ are invariant at a characteristic wavelet, $\alpha(t, X) =$ constant, which moves with an invariant speed which is determined by the value of $\mathbf{u}$ it carries. Since $\mathbf{u} = \mathbf{U}(\alpha)$, the $U_{i,X} \equiv 0$ so that (1.10) is trivially satisfied no matter what the amplitude of the u_i. If now the dependence of $\mathbf{A}$ on X is slowly varying for the wave then it is not unreasonable to suppose that in some vicinity of $X = Y$, for some time interval, a progressing wave can be locally approximated by a finite amplitude simple wave with $A(\mathbf{u}, X)$ in (1.1) replaced by $\mathbf{A}(\mathbf{u}, Y)$. The basic problem then reduces to determining how these local simple wave solutions should be enveloped to obtain a global statement for conditions in a progressing wave. Our theory is based on the implications of the relatively undistorted approximation (1.10). In the second paper a formal justification of the theory is described. This results by regarding the waves as slowly modulated simple waves with slowly changing Riemann invariants. In this paper we present a more heuristic approach and obtain the basic results with a minimum of effort.

In § 2 it is shown that at any X where (1.10) holds, to a first approximation, the relations between the $u_{i,t}$ and the u_i are identical to those in a simple wave and that the wavelets $\alpha(t, X) =$ constant are necessarily characteristic wavelets. Although these relations are identical for all high frequency progressing waves associated with this same family of characteristic wavelets, in general they cannot be formally integrated to obtain relations between the u_i which are uniformly valid for all t. The error term, although locally small, can have a cumulative effect. However, this paper is concerned with *pulses*. A pulse is that part of the wave which arrives and passes a point X before the error term in the formal integration of these approximate simple wave relations between the $u_{i,t}$ can produce a first-order effect in the relations between the u_i. In the expansions (1.3) this amounts to approximating the $\hat{\mathbf{U}}_m(\alpha, X)$ by $\hat{\mathbf{U}}_m(0, X)$; in the modulated simple wave approach it amounts to taking the Riemann invariants as functions of X only. Although the time duration of the pulse at any X is restricted, no restriction is placed on the distance travelled by the pulse (except an implicit one that the cumulative effect of frequency dispersion has not produced a first-order effect). Formulae are obtained which describe the modulating effect of stratification on both the amplitude and distortion of pulses.

In §§ 3–5 the theory established in § 2 is used to determine conditions in a large amplitude, but shockless, pulse moving under gravity towards or away from earth. Before the arrival of the pulse the atmosphere is in mechanical equilibrium with the pressure varying linearly as a function of a Lagrangian distance measure. It need not be in thermal equilibrium. The gas flow which is generated is taken to be inviscid and adiabatic. However, because of the ambient stratification, the flow is not isentropic. Two limiting cases are discussed in detail: the limit when the effect of density stratification induced by gravity dominates the effect of thermal stratification, and conversely. In both limits the flow variables can only be expressed as explicit functions of the characteristic variable α and the Lagrangian distance measure ψ. Both the formation of shocks in compression pulses and the formation of fully amplitude dispersed regions, where details of the signal which generated the pulse are forgotten, are described in detail. It is hoped that the results are applicable to the final stages of an intense atmospheric explosion where the modulating effects of ambient stratification dominate that due to radial spread of the disturbance.

In § 6 we return to the general theory of pulse propagation and show that the theory predicts the exact variation of the acceleration at any front which is moving into a uniform region. Although a knowledge of conditions at such a front are of limited applicability, it is exact. Moreover, it introduces in a natural way the various length and acceleration scales which are important in pulses where, although the amplitude is small, non-linear effects may be important. Conditions at such a front are used to discuss the precise conditions under which non-linearity and stratification may be neglected.

The theory described in this and subsequent papers is an extension of the earlier, fundamental work of Whitham (1953, 1956). He showed how the pulse approximation to (1.3) should be modified to account for the cumulative effect of locally small non-linearity. This work generalizes his results to waves of any amplitude.

2. Progressing pulses: heuristic approach

We consider high frequency progressing pulses in stratified media whose responses are described by solutions $\mathbf{u}(t, X) = (u_1, u_2, \ldots, u_n)$ to hyperbolic equations which can be written

$$\mathbf{u}_{,X} + \mathbf{A}(\mathbf{u}, X)\,\mathbf{u}_{,t} = 0. \tag{2.1}$$

Of special interest is when equations (2.1) describe conservation laws and the elements of $\mathbf{A}$ can be written

$$\mathbf{A} = \mathbf{a}_{,\mathbf{u}} \tag{2.2}$$

for some vector $\mathbf{a}(\mathbf{u}, X)$.

If equations of the dispersive form

$$\mathbf{e}_{,X} + \mathbf{B}\mathbf{e}_{,t} = \mathbf{C}, \tag{2.3}$$

where $\mathbf{B}$ and $\mathbf{C}$ are functions of $(\mathbf{e}, X)$, have static, or low frequency, solutions in the region $X \geqslant 0$ then, in this region, they too can be replaced by equations of the form (2.1). For if

$$\mathbf{e} = \mathbf{d}(X, \mathbf{u}) \tag{2.4}$$

is any solution to the ordinary differential equations

$$\mathbf{d}_{,X} = \mathbf{C}, \tag{2.5}$$

which satisfies the initial conditions that

$$\mathbf{d} = \mathbf{u} \quad \text{at} \quad X = 0, \tag{2.6}$$

then $\mathbf{u}(t, X)$ satisfies (2.1) with

$$\mathbf{A} = (\mathbf{d}_{,\mathbf{u}})^{-1}\,\mathbf{B}\mathbf{d}_{,\mathbf{u}}. \tag{2.7}$$

High-frequency progressing pulses are examples of waves in which the state variables $(u_1, u_2, \ldots, u_n)$ are *relatively undistorted* with respect to t. Quite loosely, the variables $(f_1, f_2, \ldots, f_m)$, $m \geqslant 2$, are relatively undistorted in a wave with respect to t if at all (t, X) of interest there exists a propagating surface $\alpha(t, X) =$ constant, called a wavelet, such that the magnitude of the rate of change of any f_i $(i = 1, 2, \ldots, m)$, moving with the wavelet is small compared with the magnitude of the rate of change of f_i at fixed t. If $t = T(\alpha, X)$ denotes the arrival time of the wavelet α at X, and if

$$\mathbf{F}(\alpha, X) \overset{\text{def}}{=} \mathbf{f}(t, X) \tag{2.8}$$

then
$$|F_{i,X}| \ll |f_{i,X}| \quad (i = 1, 2, \ldots, m). \tag{2.9}$$

Since, however,
$$\mathbf{F}_{,X} = \mathbf{f}_{,X} + T_{,X}\mathbf{f}_{,t} \tag{2.10}$$

in a relatively undistorted wave

$$\mathbf{f}_{,X} \simeq -T_{,X}\mathbf{f}_{,t} \tag{2.11}$$

and
$$|F_{i,X}| \ll |T_{,X} f_{i,t}| \quad (i = 1, 2, \ldots, m). \tag{2.12}$$

In particular, if $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ are relatively undistorted then, since

$$(\mathbf{A} - T_{,X}\mathbf{1})\,\mathbf{u}_{,t} = -\mathbf{U}_{,X}, \tag{2.13}$$

the *slowness* $T_{,X}$ of the wavelets must be an eigenvalue of $\mathbf{A}(\mathbf{U}, X)$. This, in turn, implies that the wavelets are characteristic surfaces. Otherwise, (2.13) would completely determine the $u_{i,t}$ as linear forms in the $U_{i,X}$ and (2.12) could not hold; only on characteristic surfaces are the normal derivatives of the u_i not uniquely determined by the U_i and their tangential derivatives $U_{i,X}$. Accordingly, in such waves

$$T_{,X} = w(\mathbf{U}, X), \tag{2.14}$$

where
$$\det|\mathbf{A}(\mathbf{u}, X) - w(\mathbf{u}, X)\,\mathbf{1}| = 0, \tag{2.15}$$

and consequently, by (2.13), $\mathbf{U}$ must satisfy the compatibility conditions

$$\mathbf{l}(\mathbf{U}, X).\,\mathbf{U}_{,X} = 0 \tag{2.16}$$

for every left eigenvector $\mathbf{l}(\mathbf{u}, X)$ of $\mathbf{A}(\mathbf{u}, X)$ corresponding to the eigenvalue $w(\mathbf{u}, X)$. In terms of $\mathbf{u} = \mathbf{U}(\alpha, X)$ and the *incremental arrival time* of wavelets

$$\Omega = T_{,\alpha}, \tag{2.17}$$

equations (2.13) read
$$(\mathbf{A} - w\mathbf{1})\,\mathbf{U}_{,\alpha} = -\Omega\mathbf{U}_{,X}, \tag{2.18}$$

while differentiating (2.14) with respect to α and using (2.17) yields

$$\Omega_{,X} = w_{,\mathbf{u}}(\mathbf{U}, X).\,\mathbf{U}_{,\alpha}. \tag{2.19}$$

Equations (2.16), (2.18) and (2.19), with w determined from (2.15), govern the exact variations of $\mathbf{U}(\alpha, X)$ and $\Omega(\alpha, X)$. In a relatively undistorted wave, to a first approximation, (2.18) is replaced by

$$(\mathbf{A} - w\mathbf{1})\,\mathbf{U}_{,\alpha} = 0. \tag{2.20}$$

We restrict attention to waves for which w is a simple root of the characteristic condition (2.15). Then, if $\mathbf{r}(\mathbf{u}, X)$ is any right eigenvector of $\mathbf{A}(\mathbf{u}, X)$ corresponding to the eigenvalue $w(\mathbf{u}, X)$, conditions (2.20) imply that to a first approximation $\mathbf{U}_{,\alpha}$ is proportional to $\mathbf{r}$, or that

$$\mathbf{U}_{,\alpha} = F_{,\alpha}\,\mathbf{r}(\mathbf{U}, X) \tag{2.21}$$

for some scalar $F(\alpha, X)$. Equations (2.21) relate the current rates of change of all components of $\mathbf{u}$ at any X to the current rate of change of any one component and the current value of $\mathbf{u}$. They are approximate relations and cannot, in general, be formally integrated to obtain relations, which are uniformly valid for all time, between the current values of $\mathbf{u}$ at X. The error term, although locally small, has a cumulative effect in time which ultimately produces a first-order contribution to the variation in $\mathbf{u}$. There are two important exceptions when the effect of the error is not significant. The first, which will be discussed in a future paper, is when the wave generates high frequency *time periodic* disturbances. Then, by a suitable choice of the constants of integration for (2.21), the error term can be shown to have zero mean value at any X so that it produces no cumulative effect. The second exceptional case, which we discuss here, is when the wave is a *pulse* which arrives and passes X over a time interval which is not long enough for the error term to produce a significant effect.

To a first approximation, conditions in a pulse can be determined by solving ordinary differential equations; for the ordinary differential relations (2.21), which hold at all X, can be formally integrated, subject to suitable initial conditions, to determine

$$\mathbf{u} = \mathbf{V}(F, X), \tag{2.22}$$

where, according to (2.21),

$$\mathbf{V}_{,F} = \mathbf{r}(\mathbf{V}, X). \tag{2.23}$$

Equation (2.16) then provides an ordinary differential equation, the *non-linear transport equation*, for the variation of F at each α = constant wavelet. In fact, (2.16) and (2.23) imply that

$$F_{,X} + D(F, X) = 0, \tag{2.24}$$

where

$$D(F, X) = \mathbf{l}(\mathbf{V}, X)\,.\,\mathbf{V}_{,X}/\mathbf{l}(\mathbf{V}, X)\,.\,\mathbf{r}(\mathbf{V}, X). \tag{2.25}$$

Once $\mathbf{V}(F, X)$ has been determined from (2.23), and $F(\alpha, X)$ from (2.24), $\mathbf{u} = \mathbf{U}(\alpha, X)$ is known and can be inserted in (2.14) to determine $t = T(\alpha, X)$. Consequently, implicit statements to determine $\mathbf{u}(t, X)$ can be obtained. If

$$f(t, X) \stackrel{\text{def}}{=} F(\alpha, X), \tag{2.26}$$

then (2.12), with $\mathbf{f} = \mathbf{u}$, together with (2.22)–(2.25) imply that a *necessary* condition that the pulse is relatively undistorted for all of the u_i is that $f(t, X)$ satisfies the high frequency conditions

$$|r_i f_{,t}| \gg w^{-1}\left|V_{i,X} - r_i\left(\frac{\mathbf{1}.\mathbf{V}_{,X}}{\mathbf{1}.\mathbf{r}}\right)\right| \quad (i = 1, 2, \ldots, n). \tag{2.27}$$

Since the input signal $f(t,0)$ can be specified, conditions (2.27) can always be satisfied at $X = 0$ (and 'by continuity' in some neighbourhood of $X = 0$).

As an illustration consider a pulse with a front $\alpha = 0$ moving with sound speed into a region where, before the arrival of the pulse, the disturbance is known. Then, since $\mathbf{u}$ is continuous at a sound front, $\mathbf{U}(0, X)$, which satisfies the ordinary differential relations (2.16), is known, and since $F(\alpha, X)$ can be normalized so that

$$F(0, X) = 0, \tag{2.28}$$

equations (2.23) can be solved subject to the initial conditions

$$\mathbf{V}(0, X) = \mathbf{U}(0, X). \tag{2.29}$$

If the wavelet $\alpha =$ constant is tagged by the time $t = \alpha$ when it passed $X = 0$, so that

$$T(\alpha, 0) = \alpha, \tag{2.30}$$

then (2.24) determines $F(\alpha, X)$ once the signal function

$$F(\alpha, 0) = \pi(\alpha) \tag{2.31}$$

has been specified. Equation (2.14) can then be solved, subject to (2.30), to give $t = T(\alpha, X)$—the arrival time at X of the wavelet α which left $X = 0$ at $t = \alpha$. Note that in such a pulse, moving in a direction of increasing X, conditions at any station $X > 0$ at the passage of the wavelet α_0 are, to a first approximation, determined by the disturbance ahead of the pulse which, essentially, induces a stratification for the pulse, and by conditions at the passage of α_0 at any other previous station, $X = 0$ say. They are independent of conditions at all precursor wavelets $0 < \alpha < \alpha_0$. In this sense, the characteristic wavelets are the carriers of information in a high-frequency pulse.

A more formal justification of the theory will be given elsewhere. Here, to illustrate its use, we apply it to two problems which involve large-amplitude non-isentropic but shockless flows of an inviscid gas.

3. Adiabatic, non-isentropic flows of an ideal gas

We consider the uni-directional flow of an inviscid gas under the action of a constant body force, $-g$, per unit mass. Let $\overline{p}$ and $\overline{\rho}$ be any constants with the dimensions of pressure and density. Let $\overline{p}p(t,\psi)$, $(\overline{p}/\overline{\rho})^{\frac{1}{2}}u(t,\psi)$ and $(\overline{p}/\overline{\rho})h(p,E)$ denote the pressure, fluid speed, and enthalpy at time $(\overline{p}/\overline{\rho})^{\frac{1}{2}}t/g$ at the particle 'ψ' which if the gas were brought to equilibrium at constant pressure $\overline{p}$ and constant density $\overline{\rho}$ would be at a distance $(\overline{p}/\overline{\rho}g)\psi$ from the particle $\psi = 0$. Then, if $E(\psi)$ denotes the entropy at the particle ψ, the equations governing uni-directional adiabatic flow are of the form (2.1) and (2.2) with

$$\mathbf{u} = \begin{pmatrix} u \\ p+\psi \end{pmatrix}, \quad \text{and} \quad \mathbf{a} = \begin{pmatrix} -h_{,p} \\ u \end{pmatrix}. \tag{3.1}$$

The density $\overline{\rho}\rho$ and temperature measure θ are given by

$$\rho^{-1} = h_{,p} \quad \text{and} \quad \theta = -h_{,E}. \tag{3.2}$$

The current co-ordinate $x(t,\psi)$ of the particle ψ satisfies the condition that

$$x_{,t} = u \quad \text{and} \quad x_{,\psi} = \rho^{-1}. \tag{3.3}$$

Equations (2.1) and (2.2), with $\mathbf{u}$ and $\mathbf{a}$ given by (3.1), are the Lagrangian statements for the changes in density and linear momentum at a particle.

We consider a pulse behind a sound front $\alpha(t,\psi) = 0$ which is moving into a region where, prior to its arrival, the gas is at rest in mechanical equilibrium with

$$u = 0 \quad \text{and} \quad p = 1-\psi_{,} = Y \quad \text{say.} \tag{3.4}$$

The gas ahead of the pulse need not, however, be in thermal equilibrium. If at the arrival of the front of the pulse the entropy at the particle ψ is $E(\psi)$ then, according to (3.2), $\rho_0(\psi)$ and $\theta_0(\psi)$ at the arrival of the pulse are given in terms of $E(\psi)$ and $Y(\psi)$ by

$$\rho_0^{-1} = h_{,p}(Y,E) \quad \text{and} \quad \theta_0 = -h_{,E}(Y,E). \tag{3.5}$$

In the pulse it is assumed that the mechanical power generated by the density variation is so large compared with the power generated by any other source, such as heat flux, that over the time it takes the pulse to pass any particle ψ the flow is adiabatic.

In (3.4), and in what follows, we take $\bar{p}$ and $\bar{\rho}$ to be the pressure and density before the arrival of the pulse at the reference particle $\psi = 0$. We also measure E so that

$$E(0) = 0. \tag{3.6}$$

The (normalized) pressure p is taken as the signal function, so that

$$F \equiv P(\alpha,\psi). \tag{3.7}$$

Then,

$$w = [-h_{,pp}(p,E)]^{\frac{1}{2}}, \quad \mathbf{r} = \begin{pmatrix} w \\ 1 \end{pmatrix}, \quad \mathbf{l} = \begin{pmatrix} 1 \\ w \end{pmatrix} \tag{3.8}$$

and

$$\mathbf{u} = \begin{pmatrix} U(P,\psi) \\ P \end{pmatrix}, \tag{3.9}$$

where

$$u = U(P,\psi) \equiv \int_{1-\psi}^{P} w(s,\psi)\,ds. \tag{3.10}$$

When (3.7)–(3.10) are inserted, the transport equation (2.24) for $P(\alpha,\psi)$ reads

$$P_{,\psi} + D(P,\psi) = 0, \tag{3.11}$$

where

$$D(P,\psi) = \frac{1}{2}\left[1 - \left.\frac{dP}{d\psi}\right|_U\right]. \tag{3.12}$$

For an ideal gas

$$h = \frac{\gamma}{\gamma-1} p^{(\gamma-1)/\gamma} \exp\left\{\frac{\gamma-1}{\gamma} E\right\} \quad \text{and} \quad \theta = \frac{\gamma-1}{\gamma} h. \tag{3.13}$$

When (3.13) is inserted, conditions (3.8)–(3.10) yield

$$w = \gamma^{-\frac{1}{2}} p^{-(\gamma+1)/2\gamma} \exp\left\{\frac{\gamma-1}{2\gamma} E\right\} = \left(\frac{\theta_0}{\gamma}\right)^{\frac{1}{2}} Y^{-1} \left(\frac{p}{Y}\right)^{-(\gamma+1)/2\gamma}, \tag{3.14}$$

and

$$u = \frac{2\gamma^{\frac{1}{2}}}{\gamma-1} [p^{(\gamma-1)/2\gamma} - Y^{(\gamma-1)/2\gamma}] \exp\left\{\frac{\gamma-1}{2\gamma} E\right\} \tag{3.15}$$

$$= \frac{2\gamma^{\frac{1}{2}}}{\gamma-1} \theta_0^{\frac{1}{2}} \left[\left(\frac{p}{Y}\right)^{(\gamma-1)/2\gamma} - 1\right], \tag{3.16}$$

where $Y = 1-\psi$ and $\theta_0(Y)$ are the pressure and temperature at the front of the pulse. The transport equation (3.11) simplifies if (Y, α), rather than (ψ, α), are used as independent variables and if

$$Z(Y,\alpha) = p/Y, \tag{3.17}$$

rather than P, is used as dependent variable. In terms of these variables equations (3.11) and (3.16) yield the equation

$$YZ_{,Y} = \frac{1}{2}\left[(1-Z) - \frac{\gamma}{\gamma-1}\frac{Y}{\theta_0}\theta_0'(Y)\, Z^{(\gamma+1)/2\gamma}\,(Z^{(\gamma-1)/2\gamma}-1)\right] \tag{3.18}$$

for the determination of $Z(Y,\alpha)$. In terms of (Y, α) the equation for the arrival time (2.14), with w given by (3.14), reads

$$YT_{,Y} = -\left(\frac{\theta_0}{\gamma}\right)^{\frac{1}{2}} Z^{-(\gamma+1)/2\gamma}. \tag{3.19}$$

If the wavelet α = constant is tagged so that it passes $\psi = 0$ at $t = \alpha$, and if at $\psi = 0$

$$p = 1+\pi(t), \tag{3.20}$$

then, (3.18) and (3.19) must be solved subject to the conditions when

$$Y = 1{:}\ Z \equiv 1+\pi(\alpha), \quad \text{and} \quad T \equiv \alpha. \tag{3.21}$$

4. Isothermal atmosphere

For the special case when the atmosphere is in thermal equilibrium before the arrival of the pulse

$$\theta_0 \equiv 1, \tag{4.1}$$

and (3.3) integrates to give

$$\rho_0 = 1-\psi = Y = e^{-x}. \tag{4.2}$$

According to (4.2) the equilibrium density ρ_0 varies exponentially in distance.† Note that when the pulse moves into a region where $0 \leqslant \psi \leqslant 1$, $(g > 0)$, which corresponds to $0 \leqslant x \leqslant \infty$, the equilibrium density ρ_0 and pressure Y are decreasing. In particular, at the 'edge of the atmosphere'

$$x = \infty, \quad \psi = 1, \quad \text{and} \quad \rho_0 = Y = 0. \tag{4.3}$$

When the pulse moves into a region where $\psi \leqslant 0$, $(g < 0)$, which corresponds to $x \leqslant 0$, the equilibrium density and pressure increase without bound.

When $\theta_0 \equiv 1$, equations (3.18) and (3.19) are easily integrated, subject to (3.20) and (3.21), to give

$$p/Y = z = [1+\pi(\alpha)\, Y^{-\frac{1}{2}}] \tag{4.4}$$

and

$$\gamma^{\frac{1}{2}}(t-\alpha) + \ln Y = G[\pi(\alpha)\, Y^{-\frac{1}{2}}] - G[\pi(\alpha)], \tag{4.5}$$

where

$$G(y) = 2\int_0^y s^{-1}\,[(1+s)^{-(\gamma+1)/2\gamma} - 1]\, ds. \tag{4.6}$$

† The distance scale has been normalized so that the scale height

$$\Delta = \bar{p}/\bar{\rho}g$$

is one unit. At the earth's surface $|\Delta|$ is of the order of 8·5 km. The unit of time is of the order of 35 sec.

In terms of the local temperature

$$\theta = z^{(\gamma-1)/2\gamma}, \tag{4.7}$$

the fluid speed

$$u = \frac{2\gamma^{\frac{1}{2}}}{\gamma-1}[\theta-1] \tag{4.8}$$

and the local Mach number

$$M = \left(\frac{\gamma p}{\rho}\right)^{-\frac{1}{2}} u = \frac{2}{\gamma-1}\theta^{-\frac{1}{2}}[\theta-1]. \tag{4.9}$$

If the *local* frequency of the pulse at the particle $\psi(=1-Y)$ is defined as

$$\omega_L = \left|2\gamma^{-\frac{1}{2}}\frac{z_{,t}}{z-1}\right| = |\gamma^{-\frac{1}{2}}[\ln(z-1)^2]_{,t}| \tag{4.10}$$

and the *natural* frequency of the gas by

$$\omega_N = z^{(\gamma+1)/2\gamma}, \tag{4.11}$$

then, the high frequency conditions (2.27) are satisfied and the approximation valid (at least up to the time when a shock passes) if

$$\omega_L \gg \omega_N. \tag{4.12}$$

According to (4.4)–(4.6) at any particle

$$z_{,t} = \pi'(\alpha)\, Y^{\frac{1}{2}}\Omega^{-1}, \tag{4.13}$$

where the incremental arrival time

$$\Omega = 1 + 2\gamma^{-\frac{1}{2}}\pi'(\alpha)\,\Phi(\pi(\alpha), Y) \tag{4.14}$$

with

$$\Phi(\pi, Y) = \pi^{-1}[(1+\pi Y^{-\frac{1}{2}})^{-(\gamma+1)/2\gamma} - (1+\pi)^{-(\gamma+1)/2\gamma}]. \tag{4.15}$$

In particular, after the passage of the front $\alpha = 0$, at which according to (4.5) and (4.2)

$$\gamma^{\frac{1}{2}}t + \ln Y = \gamma^{-\frac{1}{2}}t - x = 0, \tag{4.16}$$

(4.13)–(4.15) predict that

$$p_{,t} = \pi'(0)\, Y^{\frac{1}{2}}[1-\gamma^{-\frac{3}{2}}(\gamma+1)\,\pi'(0)\,(Y^{-\frac{1}{2}}-1)]^{-1}, \tag{4.17}$$

$$= \pi'(0)\, e^{-\frac{1}{2}x}[1-\gamma^{-\frac{3}{2}}(\gamma+1)\,\pi'(0)\,(e^{\frac{1}{2}x}-1)]^{-1}. \tag{4.18}$$

The result (4.18) is exactly the answer which is predicted by the general theory of acceleration fronts (Varley & Cumberbatch 1965). At such a front ω_L is unbounded so that condition (4.12) is trivially satisfied. (In § 6 we show that our pulse theory is *always* exact at such a front.) Since conditions at an acceleration front are, in many ways, typical of conditions in a small-amplitude high-frequency pulse, we give a brief account of the predictions of (4.18).

At a front which is moving in a direction of decreasing equilibrium pressure (increasing x and t), equation (4.18) predicts that $p_{,t}$ will always *ultimately* increase without bound if, as it passes $x = 0$,

$$p_{,t} = \pi'(0) > 0. \tag{4.19}$$

However, $p_{,t}$ is *currently* increasing at the front as it passes $x = 0$ only if

$$p_{,t} > \frac{\gamma^{\frac{3}{2}}}{\gamma+1}; \tag{4.20}$$

or, in terms of dimensional (starred) variables, $p^*_{,t*}$ is increasing at the front as it passes a station where the pressure is p^* only if

$$\frac{1}{p^*}\frac{\partial p^*}{\partial t^*} > \frac{\gamma^{\frac{3}{2}}}{\gamma+1}\left(\frac{g}{\Delta}\right)^{\frac{1}{2}}, \tag{4.21}$$

where Δ is the scale height. If condition (4.19) is satisfied, ultimately $p_{,t}$ becomes unbounded and a shock† forms at

$$x = 2\ln\left[1+\frac{\gamma^{\frac{3}{2}}}{(\gamma+1)\pi'(0)}\right]. \tag{4.22}$$

If, as the front passes $x = 0$,

$$p_{,t} = \pi'(0) < 0, \tag{4.23}$$

then, as $x \to \infty$,

$$p_{,t} = -\frac{\gamma^{\frac{3}{2}}}{\gamma+1}e^{-x}\left[1+\left(1+\frac{\gamma^{\frac{3}{2}}}{(\gamma+1)\pi'(0)}\right)e^{-\frac{1}{2}x}+O(e^{-x})\right], \tag{4.24}$$

so that the asymptotic decay in $p_{,t}$ is independent of its value, $\pi'(0)$, at any previous station although, of course, the rate of approach to this asymptotic decay law does depend on $\pi'(0)$.

At a front which is moving in a direction of increasing equilibrium pressure (decreasing x and t) even though $|p_{,t}|$ will always increase without bound as $|x| \to \infty$, a shock will form ($\Omega \to 0$) only if at $x = 0$

$$-p_{,t} > \frac{\gamma^{\frac{3}{2}}}{\gamma+1}, \tag{4.25}$$

which in terms of dimensional variables reads

$$\frac{1}{p^*}\frac{\partial p^*}{\partial t^*} > \frac{\gamma^{\frac{3}{2}}}{\gamma+1}\left(\frac{|g|}{\Delta}\right)^{\frac{1}{2}}. \tag{4.26}$$

Equations (4.4) and (4.7)–(4.9) express the flow variables p, θ, u and M as functions of the parameters (α, Y). Equation (4.5) gives t as a function of these parameters. It remains to determine

$$x = X(\alpha, Y). \tag{4.27}$$

To do this we integrate the equation

$$X_{,\alpha} = U\Omega\,(x_{,t} = u) \tag{4.28}$$

subject to the initial condition that when

$$\alpha = 0, \quad X = -\ln Y. \tag{4.29}$$

This yields

$$x+\ln Y = \frac{2\gamma^{\frac{1}{2}}}{\gamma-1}\int_0^{\alpha}[(1+\pi(s)\,Y^{-\frac{1}{2}})^{(\gamma-1)/2\gamma}-1]\,ds$$
$$+\frac{4}{\gamma-1}\int_0^{\pi(\alpha)}[(1+\eta Y^{-\frac{1}{2}})^{(\gamma-1)/2\gamma}-1]\,\Phi(\eta, Y)\,d\eta. \tag{4.30}$$

† The effect of shocks on the flow will be described in a subsequent paper.

4.1. *Small amplitude limit*

In the small amplitude limit where both

$$|\pi| \leqslant 1 \quad \text{and} \quad |\pi Y^{-\frac{1}{2}}| \leqslant 1, \tag{4.31}$$

the pressure variation is given parametrically by (4·4) as

$$p = Y[1+\pi(\alpha)\, Y^{-\frac{1}{2}}], \tag{4.32}$$

where, according to (4.5) and (4.30), to a first approximation

$$\gamma^{\frac{1}{2}}(t-\alpha)+\ln Y = -\frac{\gamma+1}{\gamma}\pi(\alpha)\,(Y^{-\frac{1}{2}}-1) \tag{4.33}$$

and $$\gamma^{\frac{1}{2}}(x+\ln Y) = Y^{-\frac{1}{2}}\left[\int_0^\alpha \pi(s)\,ds - \tfrac{1}{2}(\gamma+1)\,\gamma^{-\frac{3}{2}}\pi^2(\alpha)\,(Y^{-\frac{1}{2}}-1)\right]. \tag{4.34}$$

The temperature $$\theta = 1+\frac{\gamma-1}{2\gamma}\pi(\alpha)\, Y^{-\frac{1}{2}} \tag{4.35}$$

and the fluid speed $$u = \gamma^{-\frac{1}{2}}\pi(\alpha)\, Y^{-\frac{1}{2}}. \tag{4.36}$$

The high-frequency condition (4.12) requires that the incremental rate of change of $\pi(\alpha)$,

$$\left|\frac{d}{d\alpha}[\ln \pi^2(\alpha)]\right| = \left|2\frac{\pi'(\alpha)}{\pi(\alpha)}\right| \geqslant \gamma^{\frac{1}{2}}[1-(\gamma+1)\,\gamma^{-\frac{3}{2}}\,\pi'(\alpha)\,(Y^{-\frac{1}{2}}-1)]. \tag{4.37}$$

In particular, at $$Y = 1, \quad \text{where} \quad p = 1+\pi(t), \tag{4.38}$$

(4.38) requires that the pulse is of high frequency in the sense that

$$2\gamma^{-\frac{1}{2}}\left|\frac{\pi'(t)}{\pi(t)}\right| \geqslant 1. \tag{4.39}$$

The classical linear ray theory of geometrical optics neglects the cumulative effects of locally small non-linearity. The dominant approximation of that theory is obtained by formally replacing the right-hand sides of (4.33) and (4.34) by zeros. This theory then yields the explicit expressions

$$p = e^{-x}[1+\pi(t-\gamma^{-\frac{1}{2}}x)\, e^{\frac{1}{2}x}], \tag{4.40}$$

$$\theta = 1+\frac{\gamma-1}{2\gamma}\pi(t-\gamma^{-\frac{1}{2}}x)\, e^{\frac{1}{2}x} \tag{4.41}$$

and $$u = \gamma^{-\frac{1}{2}}\pi(t-\gamma^{-\frac{1}{2}}x)\, e^{\frac{1}{2}x} \tag{4.42}$$

for the flow variables. This theory is valid if, in addition to conditions (4.31) and (4.37) the *small rate* condition

$$(\gamma+1)\,\gamma^{-\frac{3}{2}}\,|\pi'(\alpha)\,(Y^{-\frac{1}{2}}-1)| \leqslant 1 \tag{4.43}$$

is also satisfied. The corrections to (4.40)–(4.42), given by (4.31)–(4.36), which must be made when (4.43) is not satisfied could also be obtained by using Whitham's (1956) rule.

The results (4.31)–(4.36) can also readily be obtained by a more formal procedure. If the duration of the pulse at $Y = 1$ is $(\overline{p}/\overline{\rho})^{\frac{1}{2}}\tau/|g|$ and if the flow variables

$$\mathbf{u} = \begin{pmatrix} u \\ p-Y \end{pmatrix} \tag{4.44}$$

are regarded as functions of Y and the fast characteristic variable

$$\beta = \alpha/\tau, \tag{4.45}$$

then equations (2.1) and (2.2) with **u** and **a** given by (3.1) have formal asymptotic solutions for which

$$\mathbf{u} = \tau[\mathbf{U}_0(\beta, Y) + \tau \mathbf{U}_1(\beta, Y) + \ldots], \tag{4.46}$$

while

$$\gamma^{\frac{1}{2}}(t-\alpha) + \ln Y = \tau[T_0(\beta, Y) + \tau T_1(\beta, Y) + \ldots] \tag{4.47}$$

and

$$\gamma^{\frac{1}{2}}(x + \ln Y) = \tau^2[X_0(\beta, Y) + \tau X_1(\beta, Y) + \ldots]. \tag{4.48}$$

If

$$\pi(\alpha) = \tau\pi_0(\beta), \tag{4.49}$$

then, according to (4.31)–(4.32)

$$\mathbf{U}_0 = \pi_0(\beta)\begin{pmatrix} Y^{\frac{1}{2}} \\ \gamma^{-\frac{1}{2}} Y^{-\frac{1}{2}} \end{pmatrix}, \tag{4.50}$$

while

$$T_0 = -\frac{\gamma+1}{\gamma}\pi_0(\beta)(Y^{-\frac{1}{2}}-1) \tag{4.51}$$

and

$$X_0 = Y^{-\frac{1}{2}}\left[\int_0^\beta \pi_0(\beta)\,d\beta - \tfrac{1}{2}(\gamma+1)\gamma^{-\frac{3}{2}}\pi_0^2(\beta)(Y^{-\frac{1}{2}}-1)\right]. \tag{4.52}$$

The high-frequency condition (4.37) is satisfied because as $\tau \to 0$ the left-hand side is $O(\tau^{-1})$ while the right-hand side is $O(1)$.

As an acoustic pulse moves away from earth towards the edge of the atmosphere ($Y \to 0$) the small amplitude description (4·31)–(4.36) is only valid at those particles which have not been traversed by strong shocks and at those wavelets where

$$\pi = o(Y^{\frac{1}{2}}) \quad \text{as} \quad Y \to 0. \tag{4.53}$$

If the sound front is a compression front, so that π increases for some time after the passage of the front, the time interval at any Y after the passage of the front over which the flow is shockless and of small amplitude decreases with Y until, at

$$Y = \min\left[1 + \frac{\gamma^{\frac{3}{2}}}{(\gamma+1)\pi'(\alpha)}\right]^{-2}, \tag{4.54}$$

at some instant, a shock forms. If the sound front is an expansion front, then the interval over which the flow is of small amplitude will still, ultimately, decrease as $Y \to 0$. According to (4.33) and (4.44) over this time interval as

$$Y \to 0, \quad -\pi Y^{-\frac{1}{2}} \to \frac{\gamma}{\gamma+1}(\gamma^{\frac{1}{2}}t - x) \tag{4.55}$$

and

$$p \to e^{-x}\left[1 - \frac{\gamma}{\gamma+1}(\gamma^{\frac{1}{2}}t - x)\right], \tag{4.56}$$

while

$$u \to -\frac{\gamma^{\frac{1}{2}}}{\gamma+1}(\gamma^{\frac{1}{2}}t - x). \tag{4.57}$$

Note that, asymptotically, the flow in the expansion region is independent of the detailed flow at $Y = 1$. At any x, the flow is only given by (4.56) and (4.57) until the arrival of a shock.

As an acoustic pulse moves towards the earth in a direction of increasing ambient pressure and density ($Y \to \infty$, $[x, t] \to -\infty$) the amplitude of the flow at any wavelet decays like $e^{-|x|}$ and, according to (4.37) the high-frequency approximation becomes more accurate. The level of the pressure rate will increase in the pulse only if at any particle it is large in the sense (4.21). Then, it will increase with $|x|$ until shocks form. Such shocks remain weak and can readily be analyzed by using weak shock theory, (see Varley & Cumberbatch 1966). Their main effect is to attenuate the disturbance faster than $e^{-|x|}$. If

$$|\pi'(\alpha)| \ll \frac{\gamma^{\frac{3}{2}}}{\gamma+1}, \tag{4.58}$$

or, in terms of dimensional variables, if at $Y = 1$

$$\left|\frac{1}{p^*}\frac{\partial p^*}{\partial t^*}\right| \ll \frac{\gamma^{\frac{3}{2}}}{\gamma+1}\left(\frac{|g|}{\Delta}\right)^{\frac{1}{2}}, \tag{4.59}$$

then according to (4.31)–(4.36) the linear theory gives a uniformly valid approximation to conditions in the pulse for all $Y > 1$. If

$$|\pi'(\alpha)| \gg \frac{\gamma^{\frac{3}{2}}}{\gamma+1} \tag{4.60}$$

then asymptotically as $Y \to \infty$, between shocks,

$$\pi Y^{-\frac{1}{2}} \to \frac{\gamma}{\gamma+1}[\gamma^{\frac{1}{2}}(t-t_0)-x]\,e^{-\frac{1}{2}|x|}, \tag{4.61}$$

where t_0 is constant between any two neighbouring shocks and where

$$\pi(t_0) = 0. \tag{4.62}$$

According to (4.61), (4.32), (4.35) and (4.36) the asymptotic decay in p, θ and u when (4.60) holds is independent of the detailed flow at $Y = 1$: the flow is fully amplitude dispersed.

4.2. *Large amplitude limit*

In (4.4)–(4.12) the signal function π can vary in the range

$$-1 \leqslant \pi \leqslant \infty. \tag{4.63}$$

If we define

$$\tau(t) = \gamma^{\frac{1}{2}}\pi(t)\,[1+\pi(t)]^{(\gamma+1)/2\gamma}/2\pi'(t) \tag{4.64}$$

$$= \gamma^{\frac{1}{2}}(p-1)\,p^{(\gamma+1)/2\gamma}/2p_{,t} \quad \text{at} \quad Y = 1 \tag{4.65}$$

then the high-frequency condition (4.12) at $Y = 1$ requires that

$$|\tau(t)| \ll 1. \tag{4.66}$$

Since any level of π in the range (4.63) can be attained in an arbitrarily small time without violating condition (4.66) all admissible levels of the flow variables can, in theory, be attained at $Y = 1$ without violating the high-frequency condition.

4.2.1. *Compression pulse.* In a compression pulse, whether it is moving towards or away from earth, the incremental arrival time

$$\Omega = 1+\frac{1}{\tau(\alpha)}\left[\left(\frac{1+\pi(\alpha)\,Y^{-\frac{1}{2}}}{1+\pi(\alpha)}\right)^{-(\gamma+1)/2\gamma}-1\right] \tag{4.67}$$

decreases at a wavelet. As the compression pulse steepens $\Omega \to 0$ and the high-frequency approximation becomes more accurate until a shock forms. The *limit curve*, at which

$$\Omega(\alpha, Y) = 0, \tag{4.68}$$

forms part of the curve which bounds the region of validity of a theory which neglects shocks. If a compression wavelet α_0 has not already coalesced into an existing shock before reaching $Y = Y_0$, where $\Omega(\alpha_0, Y_0) = 0$, then at Y_0 at the passage of the wavelet α_0 the fluid acceleration is unbounded and a shock begins to form. When terms which are $O(|\tau|)$ are neglected (compared with unity) the curve along which (4.68) holds is given by

$$Y^{-\frac{1}{2}} - 1 = \frac{2\gamma}{\gamma+1} \frac{1+\pi(\alpha)}{\pi(\alpha)} \tau(\alpha). \tag{4.69}$$

Between $Y = 1$ and the limit curve (4.68) the only restriction on the variations of (π, Y) is that

$$\left| \frac{\pi}{1+\pi} (Y^{-\frac{1}{2}} - 1) \right| = O(\tau), \ll 1. \tag{4.70}$$

In the small amplitude limit where $\pi = O(\tau)$, (4.70) does not restrict the variation of Y. Since, according to (4.4)

$$\frac{p}{Y} = z = (1+\pi) \left[1 + \frac{\pi}{1+\pi} (Y^{-\frac{1}{2}} - 1) \right] \tag{4.71}$$

to a first and second approximation if terms which are $O(\tau^2)$ are neglected

$$\theta = (1+\pi)^{(\gamma-1)/2\gamma} \left[1 + \frac{\gamma-1}{2\gamma} \frac{\pi}{1+\pi} (Y^{-\frac{1}{2}} - 1) \right] \tag{4.72}$$

and

$$u = \frac{2\gamma^{\frac{1}{2}}}{\gamma-1} [\theta - 1]. \tag{4.73}$$

To a first approximation the arrival time $t(\alpha, Y)$ is given by

$$\gamma^{\frac{1}{2}}(t-\alpha) + \ln Y = 2([1+\pi(\alpha)]^{-(\gamma+1)/2\gamma} - 1)(Y^{-\frac{1}{2}} - 1), \tag{4.74}$$

while

$$x + \ln Y = x_1(\alpha) + d(\pi(\alpha))(Y^{-\frac{1}{2}} - 1), \tag{4.75}$$

where

$$x = x_1(t) = \frac{2\gamma^{\frac{1}{2}}}{\gamma-1} \int_0^t ([1+\pi(s)]^{(\gamma-1)/2\gamma} - 1)\, ds \tag{4.76}$$

is the trajectory of the particle $Y = 1 (\psi = 0)$ and

$$d(\pi) = -2 \left[1 + \frac{2}{\gamma-1} (1+\pi)^{-(\gamma+1)/2\gamma} - \frac{\gamma+1}{\gamma-1} (1+\pi)^{-1/\gamma} \right]. \tag{4.77}$$

To a first approximation the terms in the square brackets in (4.71) and (4.72) may be replaced by unity so that θ and u are invariant at a characteristic wavelet, as they would be in a simple wave. However, the trajectory of the characteristic wavelet is not, in general, that of a characteristic wavelet in a simple wave. In much the same way that the effect of locally small amplitude dispersion may accumulate and produce a first-order contribution in the calculation of the flow, the effect of locally small stratification may also produce a first-order contribution in the calculation of the flow. Only when

$$|\psi| \ll 1, \tag{4.78}$$

that is only over distances which are small compared with the scale height Δ, can (4.74) and (4.75) be approximated by

$$\gamma^{\frac{1}{2}}(t-\alpha) = \psi[1+\pi(\alpha)]^{-(\gamma+1)/2\gamma} \tag{4.79}$$

and

$$x-\psi = x_1(\alpha)+\tfrac{1}{2}d(\pi(\alpha))\,\psi \tag{4.80}$$

—the exact representation of the flow in the absence of a body force and the stratification it induces.

4.2.2. *Expansion pulse.* In an expansion pulse Ω increases at a wavelet whether it is moving towards or away from earth. Conditions in such a pulse may be more complex than in a compression pulse. For, whereas in a compression pulse the high-frequency approximation improves as the wave steepens, until shocks form, in an expansion pulse the approximation may worsen until it is invalid. Then the effects of 'reflected waves' must be taken into account. For example, consider an expansion pulse, in which

$$-1 < \pi < 0 \quad \text{and} \quad 0 < \tau \ll 1, \tag{4.81}$$

moving away from the earth. Then, according to (4.10)–(4.12) and (4.67), the high-frequency approximation is valid and the effect of reflected waves is unimportant only if

$$0 \leqslant \frac{\Omega\tau}{1+(\Omega-1)\tau} \ll 1. \tag{4.82}$$

Since, according to (4.67), at any wavelet at which (4.81) hold $\Omega-1$ increases from zero at $Y=1$ to an infinite value at $Y=\pi^2$, where the pulse is fully expanded, the inequality (4.82) must at some stage be violated and the effect of reflected waves must become important.

The situation is well illustrated by considering conditions in that part of an expansion pulse which is moving away from earth, where the amplitude of

$$\Delta p = \frac{p}{Y(1+\pi)}-1 = \frac{\pi}{1+\pi}(Y^{-\frac{1}{2}}-1) \tag{4.83}$$

is small compared with unity. Note that this region, where

$$0 \leqslant -\Delta p \ll 1, \tag{4.84}$$

contains the small amplitude region neighbouring the front as well as a region neighbouring $Y=1$ where π can vary in its full range $-1<\pi\leqslant 0$. If terms which are $O(\Delta p)$ are neglected compared with unity, then

$$\Omega = 1-\frac{\gamma+1}{2\gamma}\frac{\Delta p}{\tau} \tag{4.85}$$

and the high frequency condition (4.82) requires that

$$0 \leqslant \Omega\tau = \tau-\frac{\gamma+1}{2\gamma}\Delta p \ll 1. \tag{4.86}$$

In regions where (4.84) and (4.86) hold, the flow is again described to a good approximation by (4.71)–(4.77). At any Y, Δp and π decrease as t increases and the time interval, measured from the arrival of the pulse, over which (4.84) and (4.86) are valid decreases as Y decreases. In fact, as $Y \to 0$ this time interval is arbitrarily small.

4.2.3. *Centred expansion fans*. Now, although the pulse approximation may worsen at any Y as t increases, in many situations before the approximation becomes invalid either a shock arrives or the flow becomes fully amplitude dispersed. In any fully dispersed region, to a first approximation, the flow is independent of the details of the flow at any previous time; only some gross features are remembered. This phenomena is a generalization of that which exists in an expansion simple wave when in the far field, away from the source of the disturbance, the flow approximates that produced by a centred expansion fan. In the region where (4.84) holds the flow is fully amplitude dispersed when the expression (4.85) for Ω can be approximated by

$$\Omega = -\frac{\gamma+1}{2\gamma}\frac{\Delta p}{\tau}, > 0. \tag{4.87}$$

That is when

$$1 \gg -\Delta p \gg \frac{2\gamma}{\gamma+1}\tau. \tag{4.88}$$

To describe the pattern of the fully dispersed regions it is convenient to work with

$$\lambda = 2[(1+\pi)^{-(\gamma+1)/2\gamma} - 1], = \Lambda(\alpha), \tag{4.89}$$

rather than π, and to write (4.74) as

$$(Y^{-\frac{1}{2}} - 1)\lambda = \gamma^{\frac{1}{2}}(t-\alpha) + \ln Y. \tag{4.90}$$

Then, according to (4.64), (4.83) and (4.88), in the fully dispersed region

$$\gamma^{-\frac{1}{2}}\Lambda'(\alpha)(Y^{-\frac{1}{2}} - 1) \gg 1. \tag{4.91}$$

These regions first form at wavelets $\alpha_1, \alpha_2, \ldots, \alpha_N, \ldots$ where $\Lambda'(\alpha)$ (> 0) has a local maximum. If Y_N is the least value of Y at which, to the accuracy required, the right-hand side of (4.91) is negligible compared with $\gamma^{-\frac{1}{2}}\Lambda'(\alpha_N)(Y_N^{-\frac{1}{2}} - 1)$ then a fully dispersed region begins to form at the wavelet α_N at the point (t_N, Y_N), where t_N is given in terms of α_N and Y_N by (4.90) with $\lambda = \Lambda(\alpha_N)$. This region, which is centred at (t_N, Y_N), begins to spread out around the wavelet α_N. The first approximation λ_1 to λ in this region is given by

$$(Y^{-\frac{1}{2}} - 1)\lambda_1 = \gamma^{\frac{1}{2}}(t-\alpha_N) + \ln Y. \tag{4.92}$$

The second approximation λ_2 is given by

$$(Y^{-\frac{1}{2}} - 1)\lambda_2 = \gamma^{\frac{1}{2}}(t-\alpha_N-\Delta\alpha) + \ln Y, \tag{4.93}$$

where, according to (4.89)

$$\Delta\alpha = [\lambda_1 - \Lambda(\alpha_N)]/\Lambda'(\alpha_N). \tag{4.94}$$

If $\Delta\alpha$ is eliminated, (4.93) and (4.94) imply that

$$(Y^{-\frac{1}{2}} - 1)\lambda_2 = (\gamma^{\frac{1}{2}}(t-\alpha_N) + \ln Y)(1 - [\gamma^{-\frac{1}{2}}\Lambda'(\alpha_N)(Y^{-\frac{1}{2}} - 1)]^{-1}) + \frac{\gamma^{\frac{1}{2}}\Lambda(\alpha_N)}{\Lambda'(\alpha_N)}. \tag{4.95}$$

Higher-order approximations to λ can be obtained by an obvious generalization of the scheme already applied to (4.89) and (4.90) to obtain λ_1 and λ_2. Note that λ_1 and λ_2 are obtained as explicit functions of (t, Y). (The definitions of (t_N, Y_N) can be made more precise by defining Y_N as the least value of Y for which the indicated iteration scheme converges.) The flow described by (4.92) is a generalization of that produced by a centred simple wave. To see this suppose that at

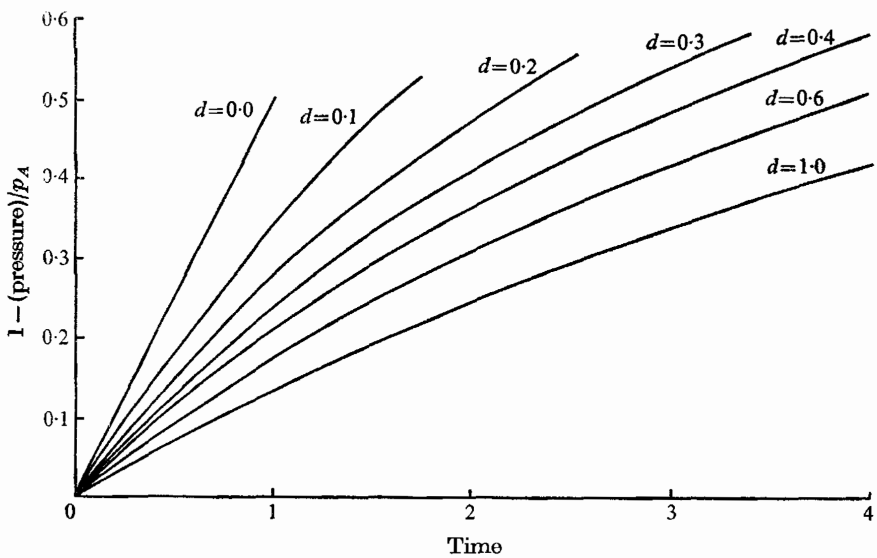

Figure 1. Pressure variations induced at particles by an acoustic expansion pulse moving away from earth.

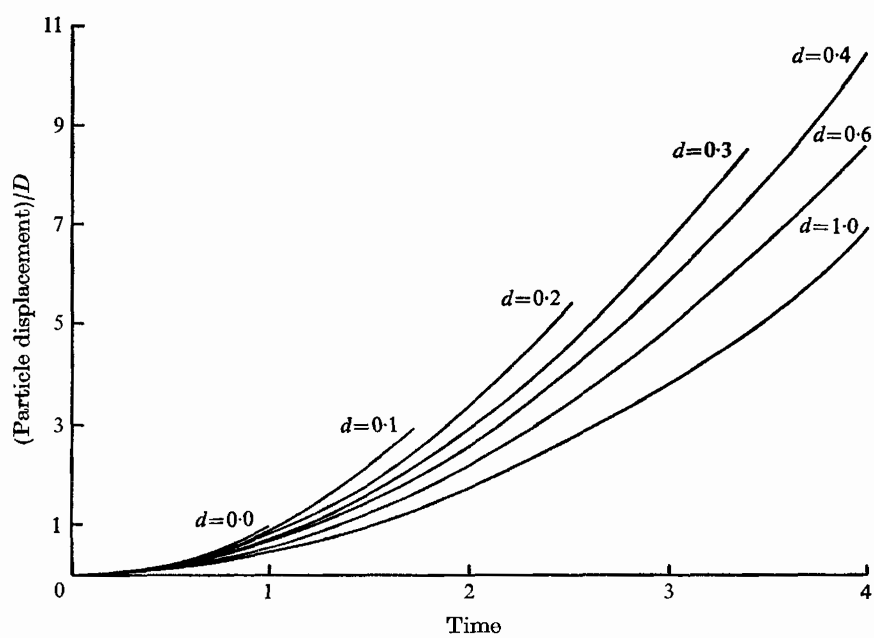

Figure 2. Particle displacements in same acoustic pulse moving away from earth.

$Y = 1$ at $t = 0$ the flow is discontinuously expanded so that $\Lambda'(0)$ is unbounded. Then, a fully dispersed region centred at $(Y, t) = (1, 0)$ forms. In this region, to a first approximation, (4.92) implies that

$$\lambda = (\gamma^{\frac{1}{2}}t + \ln Y)/(Y^{-\frac{1}{2}} - 1). \tag{4.96}$$

If $Y = 1 - \psi$, then, for $|\psi| \ll 1$, (4.96), (4.90) and (4.32) imply that

$$p = 1 + \pi = \left(\frac{\psi}{\gamma^{\frac{1}{2}}t}\right)^{2\gamma/(\gamma+1)}, \tag{4.97}$$

which is the exact expression for p in a centred simple wave. It should be noted that (4.96) is not an exact solution of the governing equations, but only the pulse approximation to a solution.

4.3. *Illustrations*

As an illustration of the results described in §4, in figures 1–4 we depict the pressure variations and displacements induced at several particles by an acoustic expansion pulse as it propagates into an isothermal atmosphere. At the reference particle $Y = 1$, where prior to the arrival of the pulse the pressure is $\overline{p}$ and the density is $\overline{\rho}$, the pressure is taken to decrease at a constant rate. Only that part of the pulse is represented where conditions are determined by what was happening at $Y = 1$ over the time interval

$$\tau = 0{\cdot}05(\overline{p}/\overline{\rho})^{\frac{1}{2}}\,|g|^{-1}, \tag{4.98}$$

when the pressure changed from $\overline{p}$ to $\frac{1}{2}\overline{p}$.

Two cases are considered. Figures 1 and 2 depict conditions at a pulse moving away from earth into a region of decreasing pressure. If $d[\overline{p}/\overline{\rho}\,|g|]$ denotes the distance measured from the initial position of the reference particle $Y = 1$, before the arrival of the pulse the pressure

$$p_A = \overline{p}e^{-d}. \tag{4.99}$$

Figure 1 shows the variation in incremental pressure at several particles as a function of time measured from the arrival of the pulse. The particles are labelled by their distances $d[\overline{p}/\overline{\rho}\,|g|]$ before the arrival of the pulse. Figure 2 shows the displacement of these particles. The time is measured in units of τ. The particle displacements are measured in units of the total displacement of the reference particle $Y = 1$ which is

$$D = 0{\cdot}001\frac{\overline{p}}{\overline{\rho}\,|g|}. \tag{4.100}$$

Figures 3 and 4 depict analogous conditions at a pulse moving towards the earth into a region of increasing pressure. Here

$$p_A = \overline{p}e^{d}, \tag{4.101}$$

while τ and D are still given by (4.98) and (4.100).

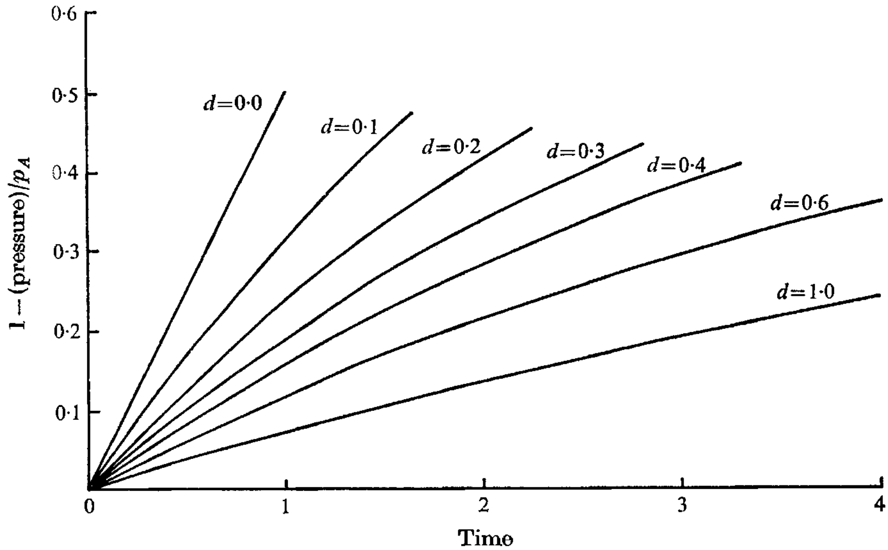

FIGURE 3. Pressure variations induced at particles by an acoustic expansion pulse moving towards earth.

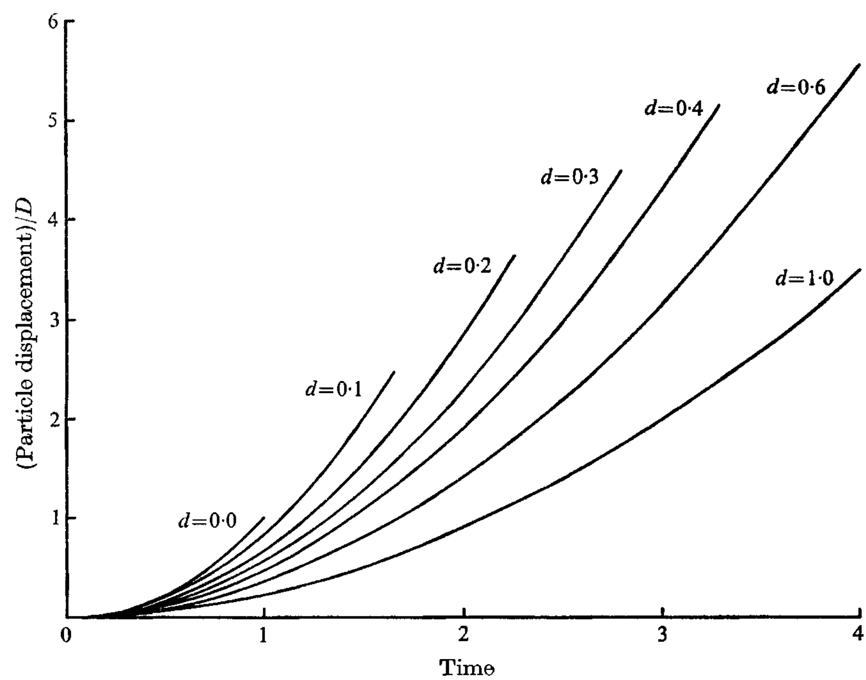

FIGURE 4. Particle displacements in same acoustic pulse moving towards earth.

5. Thermally stratified atmosphere

When the gas is not in thermal equilibrium at the arrival of the pulse and when the stratification due to thermal gradients dominates that due to body force the transport equation (3.18) can be approximated by

$$Z_{,Y} = -\frac{1}{2}\frac{\gamma}{\gamma-1}\frac{\theta_0'(Y)}{\theta_0(Y)} Z^{(\gamma+1)/2\gamma}(Z^{(\gamma-1)/2\gamma}-1). \tag{5.1}$$

This integrates to give $$z = \frac{p}{Y} = [1+\bar{\pi}(\alpha)\,\theta_0^{-\frac{1}{2}}]^{2\gamma/(\gamma-1)}, \tag{5.2}$$

where $$\bar{\pi}(\alpha) = [1+\pi(\alpha)]^{(\gamma-1)/2\gamma}-1. \tag{5.3}$$

According to (5.2) and (3.13)–(3.16) the temperature variation

$$\theta = \theta_0[1+\bar{\pi}(\alpha)\,\theta_0^{-\frac{1}{2}}], \tag{5.4}$$

and the fluid velocity $$u = \frac{2\gamma^{\frac{1}{2}}}{\gamma-1}\bar{\pi}(\alpha)\,\theta_0^{\frac{1}{2}}. \tag{5.5}$$

In (5.1)–(5.5) the ambient temperature variation $\theta_0(Y)$ is arbitrary, it is related to the ambient density variation $\rho_0(Y)$ by the ideal gas relation

$$Y = \rho_0\theta_0. \tag{5.6}$$

To interpret $\bar{\pi}(\alpha)$ note that according to (5.4) at $Y = 1$, where $\theta_0 = 1$ and $\alpha = t$, the temperature variation is given by

$$\theta = 1+\bar{\pi}(t). \tag{5.7}$$

If the pulse is not affected by a body force, (5.1) is the exact form of the transport equation (3.18) and Y in the relations (5.2) and (5.7) must be replaced by unity. Then equations (5.2), (5.4) and (5.5) express p, θ, and u at a particle in terms of $\bar{\pi}(\alpha)$ and the temperature θ_0 at that particle at the arrival of the pulse.

It remains to determine $t = T(\alpha, Y)$ and $x = X(\alpha, Y)$. It follows immediately from (3.19) and (5.2) that

$$\gamma^{\frac{1}{2}}(t-\alpha) = -\int_1^Y [Z(\alpha,s)]^{-(\gamma+1)/2\gamma}[\theta_0(s)]^{\frac{1}{2}}s^{-1}\,ds, \tag{5.8}$$

and that, in particular, the arrival time of the pulse front $\alpha = 0$ at Y is given by

$$\gamma^{\frac{1}{2}}t = -\int_1^Y [\theta_0(s)]^{\frac{1}{2}}s^{-1}\,ds. \tag{5.9}$$

To obtain $X(\alpha, Y)$ the relation (4.28), with u given by (5.5) and Ω by

$$\Omega = 1+\frac{\gamma+1}{\gamma^{\frac{1}{2}}(\gamma-1)}\bar{\pi}'(\alpha)\int_1^Y [Z(\alpha,s)]^{-1}[\theta_0(s)]^{\frac{1}{2}}s^{-1}\,ds, \tag{5.10}$$

is integrated subject to the initial data that

$$X(0, Y) = -\int_1^Y \theta_0(s)\,s^{-1}\,ds. \tag{5.11}$$

The high-frequency conditions (2.27) are satisfied if the local frequency of the pulse

$$|2\gamma^{-\frac{1}{2}}u_{,t}/u| \gg \theta_0^{-\frac{3}{2}}\theta_0' \, Y z^{(\gamma+1)/2\gamma}. \tag{5.12}$$

In the absence of body force, the factor s^{-1} in (5.8)–(5.11) and the factor Y in (5.12) must be replaced by unity. Then the constant g, which is used in §3 to define a distance and time scale, can be chosen arbitrarily. Since all expressions are homogeneous in t and Y they are form invariant under a change of scale.

According to (5.2), (5.4) and (5.5), at any Y the local maxima and minima of p, θ and u occur at the same instant at the arrival of wavelets which carry local maxima or minima of $\overline{\pi}$. The amplitude of u at any such wavelet varies like $\theta_0^{\frac{1}{4}}$ and the amplitude of $\theta-\theta_0$ varies like $\theta_0^{\frac{3}{4}}$.

6. Acceleration fronts

In §4 it was stated that the pulse approximation which was described in §2 always yields the exact variation of $\mathbf{u}_{,t}$ at any acceleration front, $\alpha = 0$ say, which is moving into an undisturbed region where $\mathbf{u} = 0$. This is easily seen by comparing the predictions of equations (2.16), (2.18) and (2.19), which are an exact restatement of equations (2.1), with the predictions of the pulse theory described by equations (2.21)–(2.25). Since at the front $\mathbf{U}_{,X} = 0$, (2.18) immediately implies that

$$\mathbf{U}_{,\alpha} = F_\alpha(X)\,\mathbf{r}(0,X) \tag{6.1}$$

for some scalar $F_\alpha(X)$. Differentiating the exact expression (2.16) with respect to α and using (6.1) to eliminate $\mathbf{U}_{,\alpha}$ then implies that $F_\alpha(X)$ satisfies the ordinary differential equation

$$\frac{dF_\alpha}{dX} = k(X)\,F_\alpha, \tag{6.2}$$

where

$$k(X) = -\frac{\mathbf{l}(0,X)\,.\,\mathbf{r}_{,X}(0,X)}{\mathbf{l}(0,X)\,.\,\mathbf{r}(0,X)}. \tag{6.3}$$

Since

$$\mathbf{u}_{,t} = \Omega^{-1}\mathbf{U}_{,\alpha}, \tag{6.4}$$

it remains to determine the variation of Ω on $\alpha = 0$. This is supplied by equation (2.19) which, using (6.1), implies that on $\alpha = 0$

$$\frac{d\Omega}{dX} = \frac{k(X)}{A(X)}F_\alpha, \tag{6.5}$$

where

$$k/A = \mathbf{r}(0,X)\,.\,w_{,\mathbf{u}}(0,X). \tag{6.6}$$

Equations (6.1), (6.2), (6.5) and (6.4) govern the exact variation of $\mathbf{u}_{,t}$ at any acceleration front. The variation of $\mathbf{u}_{,X}$ follows immediately from the fact that

$$\mathbf{u}_{,X} + w(0,x)\,\mathbf{u}_{,t} = 0. \tag{6.7}$$

To see that the pulse theory also predicts the variations described by equations (6.1)–(6.6) first note that at the front (2.21) agrees with the exact expression (6.1) with

$$F_\alpha(X) = F_{\,\alpha}(0,X). \tag{6.8}$$

In addition, if the transport equation (2.24) is differentiated with respect to α and if the expressions (2.25) and (2.23) are used to show that

$$D_{,F}(0,X) = -k(X), \tag{6.9}$$

it follows that the pulse theory also predicts that $F_{,\alpha}$ satisfies (6.2). It remains to show that the value of $T_{,\alpha}$ at the front which is predicted by pulse theory agrees with the exact value Ω. To do this note that pulse theory approximates $w(\mathbf{u}, X)$ by

$$w(\mathbf{V}, X), = \overline{w}(F, X) \quad \text{say}, \tag{6.10}$$

and takes

$$T_{,X} = \overline{w}(F, X). \tag{6.11}$$

According to (6.11) at $\alpha = 0$

$$\frac{dT_{\alpha}}{dX} = \overline{w}_{,F}(0,X)\, F_{,\alpha}. \tag{6.12}$$

The result now follows from the fact that

$$\overline{w}_{,F}(0,X) = w_{,\mathbf{u}}(0,X).\mathbf{V}_{,F} = \mathbf{r}(0,X).w_{,\mathbf{u}}(0,X) = k/A. \tag{6.13}$$

In much the same way that it has been shown that pulse theory correctly predicts the variation in the first derivatives of **u** at a front, it can be shown that, more generally, it also predicts the variations of the first non-zero derivatives of **u** no matter what their order.

Although a knowledge of conditions at an acceleration front in a stratified medium is of limited value, it is exact. Moreover, it introduces the basic length and acceleration scales which are important in the small amplitude region of a high frequency pulse. For these reasons, the predictions of the exact expressions (6.1)–(6.7) are briefly reviewed.

The variation in $F_{\alpha}(X)$, which is governed by (6.2), introduces a local length scale $|k(X)|^{-1}$ which is a measure of the local stratification of the medium. A small amplitude pulse is of high frequency at X if some associated local wavelength is small compared with $|k(X)|^{-1}$. In particular, when **A** in (2.1) is independent of X, $k \equiv 0$ and any progressing pulse moving into a uniform region is of high frequency.

To understand the significance of the local critical acceleration $A(X)$, defined by (6.3) and (6.6), it is best to write down the equation for the acceleration

$$a(X) = f_{,t} = \Omega^{-1} F_{,\alpha} \tag{6.14}$$

at the front. It follows from (6.2), (6.5) and (6.14) that

$$\frac{da}{dX} = \left(1 - \frac{a}{A}\right) ka. \tag{6.15}$$

Equations which are identical in form to (6.15) also govern the variation in strength along bi-characteristics of an acceleration front whose propagation is governed by quite general systems of quasi-linear hyperbolic equations in more than two independent variables (see Varley & Cumberbatch 1965). Equation (6.15) clearly shows the significance of $A(X)$: $|a|$ is increasing as the front passes $X = Y$ if, when $k(Y) > 0\,(< 0)$, $a(Y)/A(Y) < 1\,(> 1)$.

Since (6.15) can be replaced by a linear equation for a^{-1} it can readily be integrated to give

$$a(X) = a(0)\exp\left[\int_0^X k(s)\,ds\right][1+a(0)\,N(X)]^{-1}, \tag{6.16}$$

where
$$N(X) = \int_0^X \frac{k(r)}{A(r)}\exp\left[\int_0^r k(s)\,ds\right]dr. \tag{6.17}$$

The exact expression (6.16) can be used to indicate when either the effects of non-linearily or stratification might be neglected. If equations (2.1) are formally linearized about $\mathbf{u} = 0$, $\overline{w}_{,F} = 0$ ($|A| = \infty$) and $a = a_s$ satisfies

$$\frac{da_s}{dX} = k(X)\,a_s, \tag{6.18}$$

which integrates to give

$$a_s = a(0)\exp\left[\int_0^X k(s)\,ds\right]. \tag{6.19}$$

When $\overline{w}_{,F} \not\equiv 0$, equation (6.15) *suggests* that a_s should be a good approximation to a, at least locally, when

$$|a_s/A| \ll 1 \quad \text{(locally small accelerations)}. \tag{6.20}$$

Actually, condition (6.20) is not, by itself, sufficient to ensure that a_s is a uniformly good approximation to a. The effect of locally small non-linearity may be cumulative. If

$$\tilde{N}(Y) = \max_{0\leqslant X\leqslant Y} |N(X)| \tag{6.21}$$

then, according to (6.16) and (6.19), $a_s(X)$ is a uniformly good approximation to $a(X)$ for $0 \leqslant X \leqslant Y$ when the magnitude of the imposed acceleration

$$|a(0)| \ll \text{global critical acceleration, } \tilde{N}^{-1}$$
$$\text{(uniformly small acceleration limit)}. \tag{6.22}$$

In many situations $\tilde{N}(Y)$ grows without bound with increasing Y so that non-linear effects cannot be neglected even when (6.20) holds. Then there are, in general, two possibilities: either

$$a(0)\,N(X_c) = -1 \tag{6.23}$$

for some critical value $X = X_c$, in which case $\Omega \to 0$ and $a/a(0) \to \infty$ as $X \to X_c$; or

$$a(0)\,N(X) \to \infty \quad \text{as} \quad X \to \infty, \tag{6.24}$$

in which case
$$a \to \exp\left[\int_0^X k(s)\,ds\right]\Big/N(X) \quad \text{as} \quad X \to \infty, \tag{6.25}$$

which is independent of the imposed acceleration $a(0)$.

If there is no stratification, so that

$$k \equiv 0 \quad \text{and} \quad \overline{w}_{,F} \equiv \text{constant}, \tag{6.26}$$

then the pulse is a simple wave and $a = a_l$ satisfies

$$da_l/dX = -\overline{w}_{,F}(a_l)^2 \tag{6.27}$$

which integrates to give

$$a_l = a(0)\,[1+a(0)\,\overline{w}_{,F}X]^{-1}. \tag{6.28}$$

Here, the variation in a is completely controlled by the non-linear response of the medium. This variation can only be neglected over distances from $X = 0$ which are small compared with the shock distance $|d|$, where

$$d = [a(0)\,\overline{w}_{,F}]^{-1}. \tag{6.29}$$

When k and $\overline{w}_{,F}$ vary with X, equation (6.15) *suggests* that

$$a_l = a(0)\left[1 + a(0)\int_0^X \frac{k(s)}{A(s)}\,ds\right]^{-1}, \tag{6.30}$$

which satisfies (6.27) for varying $\overline{w}_{,F}$, should provide a good approximation to a when

$$\left|\frac{a_l}{A}\right| \gg 1 \quad \text{(locally large accelerations).} \tag{6.31}$$

Such large amplitude acceleration fronts are produced at $X = 0$ when the shock distance

$$|d| \ll |k(0)|^{-1}. \tag{6.32}$$

Conditions at such a front change significantly over distances from $X = 0$ which are small compared with $|k(0)|^{-1}$ but which are comparable with $|d|$. In this layer, where

$$|k(0)|^{-1} \gg X = O(|d|), \tag{6.33}$$

as $|a(0)/A(0)| \to \infty$, $a(X)$ is given to all approximations by (6.30). To a first approximation it is given by (6.28). If the front is compressive ($d < 0$), $a/a(0)$ becomes unbounded, to a first approximation, when

$$X = -d. \tag{6.34}$$

If the front is expansive ($d > 0$) then at the outer edge of the layer (6.33), where

$$|k(0)|^{-1} \gg X \gg d, \tag{6.35}$$

$$a \to (\overline{w}_{,F}X)^{-1} \quad \text{as} \quad |a(0)/A(0)| \to \infty, \tag{6.36}$$

so that, to a first approximation, conditions at the front are already independent of $a(0)$. More generally, when $X \gg d$, $a(X)$ at an expansion front is given by (6.25).

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REFERENCES

Varley, E. & Cumberbatch, E. 1965 *J. Inst. Maths. Applics.* **1**, 101–112.
Varley, E. & Cumberbatch, E. 1966 *J. Inst. Maths. Applics.* **2**, 133–143.
Whitham, G. B. 1953 *Comm. Pure Appl. Math.* **6**, 397–414.
Whitham, G. B. 1956 *J. Fluid Mech.* **1**, 290–318.